FUZZY GRAPH INDICES WITH APPLICATIONS TO INTERCONNECTION NETWORKS

Synopsis of the PhD Thesis

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ABSTRACT

The first part of the thesis investigates the impact of edge deletion on Zagreb indices of simple graphs, presenting results validated on few standard graphs. Moving beyond conventional models, the second section introduces the novel concepts of first, second, and hyper Zagreb indices in the context of fuzzy graphs. The study extends to Zagreb matrices and associated energies of fuzzy graphs, accompanied by the derivation of bounds for these energies. In the third part, the study progresses to fuzzy graphs of order n, defined as $G(V, \sigma, \mu)$, where $\sigma(u)$ represents vertex membership, $\mu(u, v)$ denotes the membership value of an edge, and $\mu(u)$ signifies the strength of a vertex.

The formulation of the first fuzzy Zagreb index involves the summation of products of vertex memberships and their corresponding strengths of connecting edges. A corresponding fuzzy Zagreb matrix is introduced, providing a square matrix of order n, with specific entries derived from vertex memberships and edge membership values. The investigation extends to the introduction of the Zagreb Estrada index for fuzzy graphs, accompanied by the establishment of crucial bounds. The final section bridges fuzzy graph theory with practical applications, showcasing its precision and adaptability in representing structural graphs. The fuzzy Zagreb index is analyzed within the context of interconnection networks, with a specific focus on financial mathematics. The study not only provides analytical results for the fuzzy Zagreb index but also explores its practical implications in designing efficient investment strategies.

The study highlights the heightened precision and adaptability offered by fuzzy graphs in representing structural graphs of molecules through topological indices. The applicability of various topological indices in the domain of fuzzy graphs is explored within the

broader context of graph theory. The thesis extends its scope to elucidate the utilization of fuzzy graph theory indices in interconnection networks. Analytical results for the fuzzy Zagreb index are derived, emphasizing its practical applications in interconnecting companies within the field of financial mathematics.

Moreover, the thesis extends fuzzy graph modeling to scrutinize an innovative approach to designing efficient investment strategies. The fuzzy Zagreb index emerges as a pivotal tool in this endeavor, showcasing its power and versatility in guiding strategic decisions. This comprehensive exploration establishes valuable insights at the intersection of fuzzy graph theories and their topological indices, offering a rich foundation for researchers and practitioners alike.

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CHAPTER 1

INTRODUCTION

1.1 Introduction

Graphs serve as a fundamental framework for representing the connections among various entities and data-driven relationships. They provide a concise means to articulate the underlying structure, facilitating comprehension and analysis of the behaviors inherent to the studied concepts. A topological graph index, also called a molecular descriptor, is a mathematical formula that can be applied to any graph which models some molecular structure. From this index, it is possible to analyse mathematical values and further investigate some physio-chemical properties of a molecule. Therefore, it is an efficient method in avoiding expensive and time-consuming laboratory experiments. They can be classified by the structural properties of the graphs used for their calculation. Broadly, there are five different types of topological indices defined on degree, distance, eigenvalue, matching and mixed. For instance, there is the Hosoya index, calculated by counting non-incident edges in a graph. Furthermore, the Estrada index is based on the spectrum of the graph, the Randić connectivity index, and the Zagreb group indices are calculated using the degrees of vertices, Wiener index which is based on the topological distance of vertices in the respective graph which was initially used to compare the boiling points of some alkane isomers. Widely, topological indices of graphs have theoretical and practical applications in areas such as mathematics, chemistry, physics, biology, cryptography, medicine, architecture and so on.

In graph theory, two essential components are the vertex set, representing objects within

the context, and the edge set, signifying relationships between vertices. When uncertainties surround the characterization of both vertices and edges, the concept of fuzzy graphs becomes indispensable.With advancements in large-scale integrated circuit technology, complex interconnectivity networks have become more feasible to construct. Graph theory serves as the primary tool for the construction and analysis of such networks. These interconnected subjects find clearer elucidation through the lens of graph theory and connectivity networks. The advantage of employing fuzzy graphs lies in their capacity to represent intricate relationships and uncertainties, especially when the strength of connections remains imprecise or ambiguous. The dissociation of vertex memberships and edge membership values provides a flexible framework for modeling and analyzing diverse real-world scenarios. For instance, a fuzzy graph can effectively depict the design of an interconnected network of industries, with nodes representing various companies and edges signifying correlated relationships among them. It is important to emphasize that there exist no constraints or prerequisites regarding the interdependence of vertex membership values and edge membership values. These two sets of values remain entirely autonomous, permitting their definition based on distinct criteria or considerations. This inherent flexibility opens up a wide spectrum of applications and modeling scenarios in which relationships among elements can be effectively represented with varying degrees of uncertainty and fuzziness.

Fuzzy graph indices are essential in various fields, including network theory, finance, and other complex systems, due to the need to address uncertainties and imprecise relationships. The need for fuzzy graph indices arises from the recognition that many real-world relationships are inherently fuzzy or uncertain. These indices offer a valuable means to model, analyze, and navigate complex systems where traditional graph theory may fall short in capturing the richness of interconnected relationships.

1.2 DEFINITIONS

This section contains all the basic definitions of various Zagreb indices of simple graphs, fuzzy graph, various fuzzy graph indices, fuzzy Zagreb matrices , their energies, and mathematical concepts that will be required throughout the thesis.

1.3 OBJECTIVES OF THE WORK

1. To study the characteristics of Zagreb indices and discuss the impact of edge deletion on Zagreb indices of molecule's structural graph.

2. To introduce and analyze extensions of topological indices, specifically the first and second Zagreb indices, within the context of fuzzy graphs. Investigate the properties of these indices in comparison to traditional graph theory.

3. To explore the formulation and properties of Zagreb matrices associated with fuzzy graphs. Investigate how these matrices capture essential information about graph structures and contribute to the understanding of fuzzy graph properties.

4. To investigate the application of fuzzy graph theory indices, focusing on the fuzzy Zagreb index, within interconnection networks. Explore how these indices enhance precision and adaptability in network modeling, with a particular emphasis on practical implications.

5. To demonstrate the practical applications of fuzzy graph theory indices, especially the fuzzy Zagreb index, in designing efficient investment strategies within the field of financial mathematics. To provide insights into how these indices can guide strategic decision-making.

   These objectives collectively aim to provide a focused and comprehensive exploration of fuzzy graph theory indices, addressing both theoretical aspects and practical applications in financial mathematics and network modeling.

CHAPTER 2

LITERATURE REVIEW

2.1 TOPOLOGICAL INDICES

- First Zagreb index for simple graph was introduced in 1972 by I. Gutman [1], which is among the oldest topological index.
- Randic index was introduced in 1975 by M. Randic [2].
- After this several topological indices were developed for simple graphs.
- In 2016, Wei Gao [3], found applications of Topological Indices in Anticancer Drugs.
- Topological indices of graphs have been studied widely and their applications are found in areas such as mathematics, chemistry, physics, biology, cryptography, architecture, pharmaceuticals, medical and so on [4–10].

2.2 FUZZY SETS AND GRAPHS

- The notion of Fuzzy sets and their relationships were initiated in 1965 by Zadeh [11], which displayed the significance of the ethics that lies between the binary digits 0 and 1. The ground-breaking concept of Zadeh has found many applications in many areas, including chemical industry, telecommunication, decision-making, networking, computer science, and smart city [7, 12].
- In 1973, Kaufmann [13] introduced the concept of fuzzy graph.

- Rosenfeld in 1975, [14, 15] measured the fuzzy relation on fuzzy sets and presented a new approach for fuzzy graphs. He considered some graph concepts such as a fuzzy tree, a fuzzy cycle, fuzzy bridges, etc., and some properties on fuzzy graphs are presented.

- In 1993 [16], Mordeson introduced the concept of fuzzy line graphs. Mordeson and Nair in 2000 [17], defined fuzzy graph complement, and Sunitha and Vijayakumar in 2002 [18] further studied it. In 2012, Mordeson and Mathew studied advanced topics in fuzzy graph theory and some applications of fuzzy graphs in [17, 19].

- Ghorai and Pal in 2016 [20], Mathew and Mordeson in 2017 [21] and Li et al. in 2019 [22], developed the structure of fuzzy graphs corresponding to several fuzzy graph concepts and many more authors contributed in the developments of fuzzy graph theory.

2.3 FUZZY TOPOLOGICAL INDICES

- In 2020 [23] introduced Weiner index of fuzzy graphs , the authors also discussed applications to illegal immigration networks. The study was further contributed by authors in [24–26].

- Following that very few have explored the study of fuzzy topological indices.

CHAPTER 3

ON ZAGREB INDICES OF GRAPHS WITH A DELETED EDGE

In the field of chemical graph theory and mathematical chemistry, topological indices are calculated based on the molecular graph of a chemical compound. Topological indices are used in various fields such as in the development of Quantitative Structure Activity Relationships QSARs. In Chemical Science, to study the physio-chemical properties of molecules most commonly used indices are Zagreb indices. In this chapter, we discuss the effect of an edge deletion on Zagreb indices of simple graphs, and verify the results obtained on some standard simple graphs such as P_n, C_n, S_n, K_n and $K_{m,n}$.

In chemical theory, molecular graph is a simple connected graph where nodes and edges between them are considered as atoms and chemical bonds between them, respectively. In the field of drug design, the ability to predict the effect of structural modifications is crucial. Studying the effect of edge deletion on Zagreb indices provides a systematic way to assess the impact of altering specific bonds on the molecular properties. This knowledge can guide medicinal chemists in optimizing compounds for improved pharmacological profiles.

The chapter includes many results in the form of theorems and illustrations. One of the important theorem proved in the chapter is stated below

Theorem 3.0.1. *If $G(V, E)$ is a simple graph and $e \in E$ is an edge incident to vertices u_1 and u_2 of the graph $G(V, E)$. Let N(u_1)=$\{ u_2, v_1, v_2, \ldots, v_k\}$ and N(u_2)=$\{ u_1, w_1, w_2, \ldots, w_l\}$, then the effect of edge deletion on the multiplicative-sum*

Zagreb index is given by

$$\frac{\prod_1^*(G-\{e\})}{\prod_1^*(G)} = \prod_{i=1}^{k}\left[1-\frac{1}{d_{u_1}+d_{v_i}}\right].\prod_{j=1}^{l}\left[1-\frac{1}{d_{u_2}+d_{w_j}}\right].\left[\frac{1}{d_{u_1}+d_{u_2}}\right] \tag{3.1}$$

CHAPTER 4

FUZZY ZAGREB INDICES

Topological indices M_1/M_2 known as first/second Zagreb indices are defined as the sum of the sum/product of degrees of pairs of adjacent vertices of a simple graph. These indices and their properties have been studied in detail under chemical graph theory. In this chapter we have introduced the fuzzy Zagreb first index, fuzzy Zagreb second index, hyper Zagreb index. The Zagreb matrices and the associated Zagreb energies of fuzzy graphs. Some bounds for these energies are obtained and illustrations for the results are included.
The fuzzy Zagreb first index has been introduced as follows:

Definition 4.0.1. The fuzzy Zagreb first index of G=(σ, μ) is defined as

$$FM_1(G) = \sum_{uv\in\mu^*}[\sigma(u)\mu(u)+\sigma(v)\mu(v)] \tag{4.1}$$

Using the newly introduced indices, we have stated and proved results on bounds of these indices. One such theorem is stated below:

Theorem 4.0.2. *Let G=(V, σ, μ) be a fuzzy graph with $|V| = n$ vertices. Let FZ be the corresponding fuzzy Zagreb matrix with eigen values $\xi_1,\ \xi_2,\ \ldots\ ,\ \xi_n$ and FZE is its*

fuzzy Zagreb energy then

$$\sqrt{2(FHM) + n(n-1)|det(FZ)|^{\frac{2}{n}}} \leq FZE \leq \sqrt{2n.(FHM)} \tag{4.2}$$

The chapter includes the studies done on Zagreb matrices and fuzzy Zagreb energy in the form of theorems and illustrations.

CHAPTER 5

FUZZY ZAGREB ESTRADA INDEX

Let $G(V, \sigma, \mu)$ be a fuzzy graph of order n, where $\sigma(u)$ is the vertex membership, $\mu(u, v)$ is the edge-strength and $\mu(u)$ is the strength of vertex. The first fuzzy Zagreb index is the sum of $\sigma(u_i)\mu(u_i) + \sigma(u_j)\mu(u_j)$ where $u_iu_j \in \mu^*$ and the corresponding fuzzy Zagreb matrix is the square matrix of order n whose $(i, j)^{th}$ entry whenever $i \neq j$, is $\sigma(u_i)\mu(u_i) + \sigma(u_j)\mu(u_j)$ and zero otherwise. In this chapter we introduce the Zagreb Estrada index of fuzzy graphs and establish some bounds for it.

Definition 5.0.1. Fuzzy Zagreb Estrada index of $G(V, \sigma, \mu)$ is defined as

$$FZEE = \sum_{i=1}^{n} e^{\xi_i} \tag{5.1}$$

where $\xi_1, \xi_2, \ldots, \xi_n$ are the eigenvalues of fuzzy Zagreb matrix *FZ*.

This chapter includes studies on bounds of the fuzzy Zagreb Estrada index and ver-

ification of the results on fuzzy graphs.

CHAPTER 6

APPLICATIONS TO INTERCONNECTION NETWORKS

In this chapter we study the utilization of fuzzy graph indices within interconnection network of financial sectors. We construct multidimensional fuzzy graphs to conduct a comprehensive sector analysis using the nine critical parameters which include: dividend paid, return on capital employed (ROCE), earnings per share (EPS), profit growth, debt-to-equity ratio, price-to-book ratio (P/B), earning yield, operating profit margin (OPM), and price-to-earnings ratio (P/E). Analysis of the real data of top five companies in various sectors such as Banking, IT, Electronics, Metal, and FMCG has been studied.

Once the investment sector is identified, fuzzy graphs of the inter connectivity network of the sector has been constructed. The nine distinct parameters are considered as nodes and the edge membership values are derived by calculating the rank correlation coefficient between these parameters. Extensive analysis of these fuzzy graphs using fuzzy Zagreb Index has been done and we drew the conclusion for the investment opportunity in financial markets. In the study, we observed that more the value of fuzzy Zagreb index of the company, better is the investment opportunity.

In addition to these theoretical advancements, we perform a comprehensive comparative study of various fuzzy topological indices across different investment sectors and individual banks. By calculating indices such as the fuzzy Zagreb index (FM), fuzzy Zagreb energy (FZE), fuzzy hyper Zagreb index (FHM), and fuzzy Zagreb Estrada index (FZEE), the analysis reveals significant insights into the structural dynamics, connectivity, and stability of financial entities.

Hence, we conclude that fuzzy Zagreb indices could be part of a broader analysis that incorporates financial and economic factors to make more informed investment decisions.

CHAPTER 7

SUMMARY AND CONCLUSIONS

The thesis carryout thorough exploration of Zagreb indices, fuzzy graphs, and their applications in diverse fields. The journey begins with an investigation into the impact of edge deletion on Zagreb indices of simple graphs, providing validated results on standard graphs. Beyond conventional models, the second section introduces novel concepts of first, second, and hyper Zagreb indices within the context of fuzzy graphs, extending the study to Zagreb matrices and associated energies.

The third part progresses to fuzzy graphs of order n. The formulation of the first fuzzy Zagreb index involves the summation of products of vertex memberships and their corresponding strengths of connecting edges. A fuzzy Zagreb matrix is introduced, providing a square matrix of order n. The investigation extends to the introduction of the Zagreb Estrada index for fuzzy graphs, establishing crucial bounds.

The thesis bridges fuzzy graph theory with practical applications, showcasing its precision and adaptability in representing structural graphs. Specifically, the fuzzy Zagreb index is analyzed within the context of interconnection networks, with a focus on financial mathematics. The study not only provides analytical results for the fuzzy Zagreb index but also explores its practical implications in designing efficient investment strategies.

Departing from conventional models, the thesis highlights the heightened precision and adaptability offered by fuzzy graphs in representing structural graphs of molecules

through topological indices. Various topological indices are explored within the broader context of graph theory, emphasizing their theoretical and practical applications in diverse fields.

The literature review provides a comprehensive overview of topological indices, fuzzy sets and graphs, and fuzzy topological indices, setting the stage for the subsequent chapters. Definitions and objectives are clearly outlined, guiding the reader through the subsequent sections.

The research objectives are meticulously addressed in subsequent chapters, covering extensions and analyses of fuzzy graph indices, development and analysis of fuzzy graph matrices, utilization in interconnection networks, and practical applications in financial mathematics. The results are presented through theorems, definitions, and illustrations, offering a rich foundation for further research.

In conclusion, this thesis contributes significantly to the understanding of Zagreb indices, fuzzy graphs, and their applications. The comprehensive exploration not only advances theoretical knowledge but also offers practical insights for researchers and practitioners alike. The interdisciplinary approach, particularly in the realms of financial mathematics and structural graph representation, opens up new avenues for future research and applications. Overall, the thesis successfully combines theoretical rigor with practical relevance, enriching the field of fuzzy graph theories and their topological indices.

CHAPTER 8

FUTURE WORK

- Few more topological indices can be introduced for fuzzy graphs.
- The need for fuzzy graph indices arises from the recognition that many real-world relationships are inherently fuzzy or uncertain. These indices offer a valuable means to model, analyze, and navigate complex systems where traditional graph theory may fall short in capturing the richness of interconnected relationships.
- Fuzzy indices and their applications are versatile and can be found in various fields due to their ability to handle uncertainty and imprecision. Potential applications could be found in the fields of Finance, Healthcare, Environmental Science, Supply chain management etc.
- As technology advances and our understanding of complex systems deepens, the future of fuzzy graph theory in financial networks holds exciting possibilities for more sophisticated modeling, analysis, and decision-making. Researchers can contribute to the field by addressing these challenges and exploring innovative applications across various domains.

PUBLICATIONS

1. M.Kale, Minirani S. *'On Multiplicative Zagreb Indices of Graphs with a Deleted Edge'*, International Conference proceedings, 2019.

2. M.Kale, Minirani S. *'On Zagreb indices of graphs with a deleted edge'*. Annals of Pure and Applied Mathematics, Vol. 21, No. 1, pp. 1-14, 2020. [Publons indexed (currently known as Web of Science)]

3. M.Kale, Minirani S. *'Some Bounds for Fuzzy Zagreb Energies'*, International Conference Proceedings, 2020.

4. M.Kale, Minirani S. *'Fuzzy Zagreb Indices and some Bounds for Fuzzy Zagreb Energies'*, International Journal of Analysis and Applications, Vol.19, No.2, pp.252-263, 2021. [Web of Science, Emerging Sources Citation, Scopus indexed]

5. M.Kale, Minirani S. *'Bounds for Fuzzy Zagreb Estrada Index'*, Communications in Combinatorics and Optimization, Vol. 8.1, pp.153-160, 2023. [Web of Science, Emerging Sources Citation, Scopus indexed]

6. M.Kale, Minirani S. *'Fuzzy Zagreb Index and Its Application in Financial Interconnection Networks'*, Turkish Journal of Computer and Mathematics Education, Vol.11, No. 03, pp.2570-2581, 2020. [Scopus indexed]

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